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The Global Element Method Applied to a Harmonic Mixed Boundary Value Problem

J. A. HENDRY* AND L. M. DELVES†

†*Department of Computational and Statistical Science, University of Liverpool Liverpool L69 3BX England*; and **Computer Centre, University of Birmingham, Birmingham B15 2TT, England*

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We demonstrate the use of the recently developed Global Element Method (Delves and Hall, *J. Inst. Math. Appl.* **23** (1979), 223–234) by applying it to the two dimensional singular boundary value problem introduced by Motz [*Quart. Appl. Math.* 4 (1946), 371–377.] The results obtained converge extremely rapidly even in the neighborhood of the singularity and suggest that the method is capable of efficiently handling singular problems.

1. INTRODUCTION

Variational methods (or the closely related Galerkin methods) have become popular for the numerical solution of differential equations. Two distinct approaches have developed, based respectively on a global (global variational method (GVM), see [10] and local (finite element method (FEM), see [14] choice of trial functions. In practice, for sufficiently smooth problems, both approaches produce satisfactory results.

Each approach has its advantages. The GVM is attractive since it is possible to achieve very fast convergence rates [3] and it is best adapted to relatively simple regions (e.g. circles, squares) for which a natural choice of trial functions can be made. A FEM, however, is well suited for an irregularly shaped region.

Both the GVM and FEM, however, give poor convergence rates for problems containing non-polynomial behaviour. It is possible to improve matters by including in the trial function either special core terms (GVM and FEM) or singular elements (FEM), to represent the non-polynomial behaviour, but a certain amount of caution must be exercised since the inclusion of too many core terms can lead to ill-conditioned matrices due to the higher core terms being themselves approximated by the polynomial terms [5].

Recently, there has been proposed [2], a new variational approach, the global element method (GEM), which attempts to retain the flexibility of the FEM for awkward shaped regions and the high convergence rate of the GVM. In the GEM, the region is split into a (small) number of subregions, these being chosen from the geometry of the region or from the anticipated different solution behaviours in the various parts of the region. Within each subregion, a suitable expansion set is chosen,

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continuity between each subregion being imposed implicitly by the variational functional (as opposed to the explicit imposition in a conventional FEM), and convergence is obtained by increasing the number of functions in each region (as in the GVM). With this approach terms describing non-polynomial behaviour can be systematically and straightforwardly included by using them as an expansion basis in a subregion containing the singularity. Moreover, the GEM as formulated in [2] permits the relaxation of the essential boundary conditions in the trial functions. The functional used in the GEM is somewhat similar to that previously considered in nuclear engineering [20, 21]. However, the GEM differs in the treatment within each subregion and in the implicit treatment of the essential boundary conditions. The GEM can also be considered as a special case of a class of recently proposed non-conforming mixed methods [1], although the derivation and subsequent choice of expansion set are rather different.

The GEM has previously been used [8] to obtain the solution of the one-dimensional Schrodinger equation with a discontinuous potential. Very good results were obtained together with a fast (exponential) convergence rate. In this paper, the GEM is applied to the two-dimensional harmonic mixed boundary value problem of Motz [12]. This problem exhibits singular behaviour and has previously been tackled by a variety of methods. We note here attempts using finite differences [12, 18]; finite elements with the inclusion of either special singular functions [11] or singular elements [15]; dual series methods [16, 17]; and conformal transformation methods [19, 13].

The conformal transformation methods (CTM) are of particular importance since they yield (subject to the determination of the special functions involved) the exact solution to the problem.

In Section 2, the Motz problem and the GEM approach to the solution are outlined while section 3 describes the results obtained. Finally section 4 contains some further comments on the GEM.

2. Description of the Problem and the Method

2.1. *The Problem*

The problem considered is that first introduced by Motz [12] and can be presented in the form (see figure 1 and [16])

$$\nabla^2 U = 0 \qquad \text{in} \quad R: -1 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1 \tag{2.1a}$$

subject to the boundary conditions

$$\begin{aligned} U &= 0 && \text{on } AO \\ U &= 500 && \text{on } BC \\ \frac{\partial U}{\partial n} &= 0 && \text{on } OB,\ AD,\ CD \end{aligned} \tag{2.1b}$$

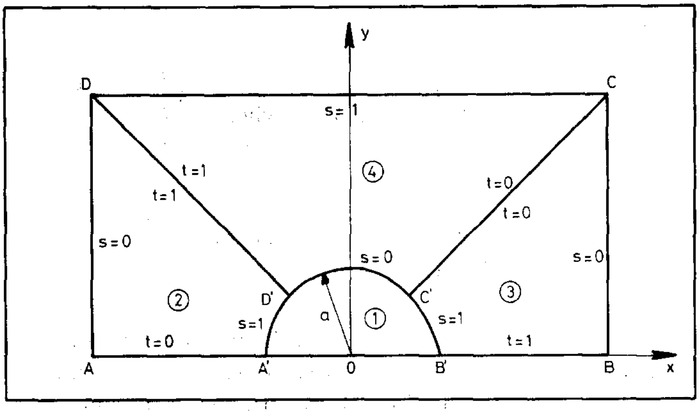

FIG. 1. The region $ABCD$ over which Laplace's equation is to be solved. The subregions used in the GEM are shown, together with the (s, t) parameterisation used in the blending function map.

where $\partial/\partial n$ is the derivative in the direction of the (outward) pointing normal and the normalisation $U = 500$ on BC has become conventional for this problem.

2.2. *The GEM Applied to the Motz Problem*

One of the motivations for the introduction of the GEM was to permit suitable trial functions to be used near a given point without these trial functions "leaking" into adjoining areas (as in a conventional GVM or FEM). In [12] it is shown that near the origin O, the exact solution to problem (2.1) has singularities in its derivatives and can be expanded, in terms of polar coordinates r, Θ centered on O, as

$$U = \sum_{i=1} \alpha_i r^{(2i-1)/2} \cos\left(\left(\frac{2i-1}{2}\right)\Theta\right) \tag{2.2}$$

the coefficients α_i being unknowns.

We therefore choose to implement the GEM with the 4 subregions labelled in figure 1, region (1) being a semi circle of radius a centred on 0. The GEM then replaces problem (2.1) by the coupled set of problems in each of the four subregions R_i $(i = 1, 4)$

$$\nabla^2 U^{(\ell)} = 0 \qquad \text{in region } R_\ell \quad \ell = 1, 2, 3, 4 \tag{2.3a}$$

subject to the boundary conditions

$$\begin{array}{llll} U^{(1)} = 0 & \text{on } A'O & U^{(2)} = 0 & \text{on } AA' \\ \dfrac{\partial U^{(1)}}{\partial n_1} = 0 & \text{on } OB' & \dfrac{\partial U^{(2)}}{\partial n_2} = 0 & \text{on } AD \\ U^{(3)} = 500 & \text{on } BC & \dfrac{\partial U^{(4)}}{\partial n_4} = 0 & \text{on } CD \\ \dfrac{\partial U^{(3)}}{\partial n_3} = 0 & \text{on } B'B & & \end{array} \tag{2.3b}$$

and the interface conditions

$$\left.\begin{aligned} U^{(1)} &= U^{(\ell)} \\ \frac{\partial U^{(1)}}{\partial n_1} &= -\frac{\partial U^{(\ell)}}{\partial n_\ell} \end{aligned}\right\} \text{ on } \Gamma_{1\ell} \quad \text{where} \quad \left\{\begin{aligned} \Gamma_{12} &\equiv A'D' \\ \Gamma_{13} &\equiv B'C' \\ \Gamma_{14} &\equiv C'D' \end{aligned}\right.$$

$$\left.\begin{aligned} U^{(4)} &= U^{(\ell)} \\ \frac{\partial U^{(4)}}{\partial n_4} &= -\frac{\partial U^{(\ell)}}{\partial n_\ell} \end{aligned}\right\} \text{ on } \Gamma_{4\ell} \quad \text{where} \quad \left\{\begin{aligned} \Gamma_{42} &\equiv D'D \\ \Gamma_{43} &\equiv C'C \end{aligned}\right. \tag{2.3c}$$

In equations (2.3b) and (2.3c), $\partial/\partial n_\ell$ represents the normal derivative in the outward direction for the appropriate region. In [2] it is shown that the unique function satisfying eqs. (2.3) is the unique solution of the original problem (2.1) while using theorem 4 of [2] it can then be shown that the functional,

$$\begin{aligned} J(w^{(\ell)}) \equiv{} & \sum_{\ell=1}^{4} \int_{R_\ell}\!\!\int (w_x^{(\ell)2} + w_y^{(\ell)2})\, dx\, dy - 2\int_{AA'} w^{(2)} \frac{\partial w^{(2)}}{\partial n_2}\, ds \\ & - 2\int_{A'O} w^{(1)} \frac{\partial w^{(1)}}{\partial n_1}\, ds - 2\int_{BC} (w^{(3)} - 500) \frac{\partial w^{(3)}}{\partial n_3}\, ds \\ & - \sum_{\ell=2}^{4} \int_{\Gamma_{1\ell}} (w^{(\ell)} - w^{(1)}) \left(\frac{\partial w^{(\ell)}}{\partial n_\ell} - \frac{\partial w^{(1)}}{\partial n_1}\right) ds \\ & - \sum_{\ell=2}^{3} \int_{\Gamma_{4\ell}} (w^{(4)} - w^{(\ell)}) \left(\frac{\partial w^{(4)}}{\partial n_4} - \frac{\partial w^{(\ell)}}{\partial n_\ell}\right) ds \end{aligned} \tag{2.4}$$

is stationary about the true solution of equation (2.3) without requiring that the trial functions $w^{(\ell)}$ should satisfy either the boundary conditions (2.3b) or the interface conditions (2.3c).

We introduce suitable expansion sets $\{h^{(\ell)}\}$ in each region

$$w^{(\ell)} = \sum_{i=1}^{N_\ell} \alpha_i^{(\ell)} h_i^{(\ell)} \tag{2.5}$$

Then inserting (2.5) into functional (2.4) and finding the stationary value (w.r.t. $\alpha_i^{(\ell)}$) leads to the symmetric (4×4) block matrix equation for the variational parameters $\alpha_i^{(\ell)}$,

$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{B}_{12} & \mathbf{B}_{13} & \mathbf{B}_{14} \\ \mathbf{B}_{12}^T & \mathbf{H}_{22} & \mathbf{0} & \mathbf{B}_{24} \\ \mathbf{B}_{13}^T & \mathbf{0} & \mathbf{H}_{33} & \mathbf{B}_{34} \\ \mathbf{B}_{14}^T & \mathbf{B}_{24}^T & \mathbf{B}_{34}^T & \mathbf{H}_{44} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \boldsymbol{\alpha}_3 \\ \boldsymbol{\alpha}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{g}_3 \\ \mathbf{0} \end{bmatrix} \tag{2.6}$$

Some typical matrix elements in equation (2.6) are

$$\mathbf{H}_{11} \equiv \int_{R_1}\int (h_{ix}^{(1)} h_{jx}^{(1)} + h_{iy}^{(1)} h_{jy}^{(1)})\, dx\, dy - \int_{A'O} \left(h_i^{(1)} \frac{\partial h_j^{(1)}}{\partial n_1} + \frac{\partial h_i^{(1)}}{\partial n_1} h_j^{(1)} \right) ds$$

$$- \frac{1}{2} \sum_{\ell=2}^{4} \int_{\Gamma_{1\ell}} \left(h_i^{(1)} \frac{\partial h_j^{(1)}}{\partial n_1} + \frac{\partial h_i^{(1)}}{\partial n_1} h_j^{(1)} \right) ds \qquad i, j = 1, \dots, N_1$$

$$\mathbf{B}_{1\ell} = \frac{1}{2} \int_{\Gamma_{1\ell}} \left(h_i^{(1)} \frac{\partial h_j^{(\ell)}}{\partial n_\ell} + \frac{\partial h_i^{(1)}}{\partial n_1} h_j^{(\ell)} \right) ds \qquad \begin{array}{l} i = 1, \dots, N_1 \quad j = 1, \dots, N_\ell \\ l = 2, 3, 4 \end{array} \qquad (2.7)$$

$$\mathbf{g}_3 \equiv -500 \int_{BC} \frac{\partial h_i^{(3)}}{\partial n_3} ds \qquad i = 1, \dots, N_3$$

$$\boldsymbol{\alpha}_\ell \equiv \alpha_i^{(\ell)} \qquad i = 1, \dots, N_\ell, \quad \ell = 1, \dots, 4$$

$\mathbf{0}$ = zero matrix or vector of appropriate size.

Similar definitions apply for the remaining quantities in equation (2.6).

The matrices $\mathbf{H}_{ii}$ which appear in (2.6) are square. However the interface matrices. $\mathbf{B}_{ij}$ are not necessarily square since we are free to use a different number of trial functions in each subregion.

Equation (2.7) show that a GEM calculation is very similar in form to a conventional Rayleigh–Ritz calculation. Indeed, the latter method can be interpreted as a single element GEM with the trial functions chosen to explicitly satisfy the Dirichlet boundary conditions.

2.3. *Choice of Trial Functions*

In each subregion R_ℓ, $l = 1, \dots, 4$, we must choose a suitable set of trial functions. As indicated earlier, in R_1 it is natural to make the orthogonal expansion

$$w^{(1)} = \sum_{i=1}^{N_1} \alpha_i^{(1)} r^{(2i-1)/2} \cos((2i - 1)/2\Theta) \qquad (2.8)$$

With this choice of trial function, each term of expansion (2.8) exactly satisfies the differential equation in R_1 and the boundary conditions along $A'O$, OB'.

For the remaining regions, we follow the construction suggested in [2] section 5, using a blending function map [7] to map the four-sided subregions R_2, R_3, R_4 onto the unit square $0 \leqslant s, t \leqslant 1$.

Having transformed each of subregions R_2, R_3, R_4, it is then convenient to use an expansion of the form

$$w^{(\ell)} = \sum_{i,j=1}^{N_\ell} \alpha_{ij}^{(\ell)} P_i(s)\, P_j(t) \qquad \ell = 2, 3, 4 \qquad (2.9)$$

where the $P_i(s)$ are suitable polynomials in each subregion, which for reasons of stability (see [10]) we take to be the orthogonal polynomials

$$P_i(s) \equiv P_{i-1}^{(0,0)}(2s - 1) \tag{2.10}$$

where $P_k^{(0,0)}$ is the Legendre polynomial of degree k. With this choice of trial function, the boundary conditions on AA' (subregion 2) and BC (subregion 3) are not satisfied exactly.

None of the interface conditions are, of course, satisfied explicitly.

3. Results

The results presented in this paper have been obtained using a semicircle of radius $a = 0.5$ for R_1; experiments showed that this value appeared to give the optimal accuracy both in the solution within each subregion and in the reproduction of the interface conditions.

The integrals required in the matrix elements (2.7) were computed by a Gauss–Legendre rule, a product of two such rules being used for the double integrals. By exploiting the product nature of the trial function in R_2, R_3, R_4 a reasonably efficient method can be developed for the double integrals in these subregions. The double integrals in R_1 were done analytically since they are very simple in form. It was found that stable results were produced when a quadrature rule of sufficiently high order was used.

For convenience, in (2.8) and (2.9) we chose

$$N_\ell = N \qquad \ell = 1,\dots,4$$

and thus there were a total of $3N^2 + N$ trial functions present. Figure 2 shows the convergence of the first six variational parameters $\alpha_i^{(1)}$ of eq. (2.8) to their exact values (from [13]) as a function of N. Here the relative error

$$E_i(N) = \left| \frac{\alpha_i^{(1)} - \alpha_i}{\alpha_i} \right|$$

has been plotted. The accuracy achieved for each parameter for a given N gradually decreases with i, the final results ($N = 9$) for $\alpha_1^{(1)}$ and $\alpha_6^{(1)}$ being accurate to about 6 and 2 figures respectively. The points are well fitted by the parallel straight lines shown indicating, from the log-linear scale used, a convergence w.r.t. N of the form

$$E_i(N) = CA^N, \qquad C \text{ constant}; \quad 0 < A < 1 \tag{3.1}$$

The ability to achieve such exponentially fast convergence appears to be a major advantage of the GEM. As might be expected from figure 2, good results are obtained for the solution at points close to O. Table 1 shows the convergence of the GEM

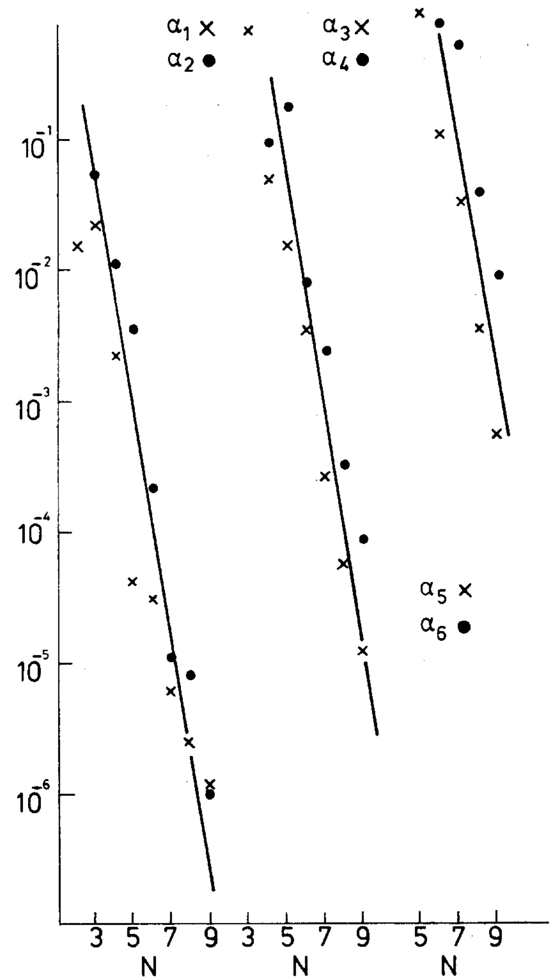

FIG. 2. Convergence of the variational coefficients in R_1 to the exact values against N.

results at a selection of points in R_1. Clearly as N increases, satisfactory convergence is obtained, the final converged results being consistent with the exact solution [19] to the number of figures quoted.

Similarly, table 2 shows the results obtained at a set of points remote from O. These points require the solution in each of R_2, R_3, R_4 and hence give a representative sample of the convergence in each subregion. (Note that where a point lies on the interface of two subregions, the average value of the solution in each subregion is given). Again satisfactory convergence is obtained to the exact results (where available). Table 2 also indicates that the GEM is successfully reproducing the boundary condition on BC ($x = 1$) as N increases. The results of tables 1 and 2 are typical of the agreement found with the exact results throughout the region.

TABLE 1

Convergence of the solution at points (x, y) close to 0[a]

N \ (x, y)	$(-1/14, 1/28)$	$(-1/28, 1/28)$	$(0, 1/28)$	$(1/28, 1/28)$	$(1/14, 1/28)$
2	30.029	37.149	54.228	80.693	103.380
3	25.444	34.375	54.337	85.403	114.115
4	24.879	33.679	53.311	83.854	112.095
5	24.805	33.590	53.187	83.676	111.875
6	24.809	33.593	53.188	83.674	111.868
7	24.808	33.592	53.186	83.672	111.865
8	24.808	33.592	53.186	83.671	111.865
9	24.808	33.592	53.186	83.671	111.865
Whiteman & Papamichael	24.81	33.59	53.19	83.67	111.86
Morley	24.74	33.53	53.09	83.46	111.58

[a] Also shown are the results of Whiteman and Papamichael [19] and Morley [11].

TABLE 2

Convergence of the solution at points (x, y) remote from 0[a]

N \ (x, y)	$(-1, 4/7)$	$(-4/7, 4/7)$	$(0, 4/7)$	$(4/7, 4/7)$	$(1, 4/7)$
2	113.615	60.817	216.368	368.190	583.626
3	63.608	87.664	181.166	345.406	493.844
4	72.235	89.421	185.426	356.563	499.647
5	72.183	89.679	183.750	356.689	500.236
6	72.251	89.785	183.969	356.669	499.981
7	72.222	89.805	183.910	356.672	499.993
8	72.223	89.803	183.911	356.677	499.999
9	72.221	89.803	183.908	356.677	500.000
Whiteman & Papamichael	—	89.80	183.91	356.68	500

[a] Also shown are the results of [19].

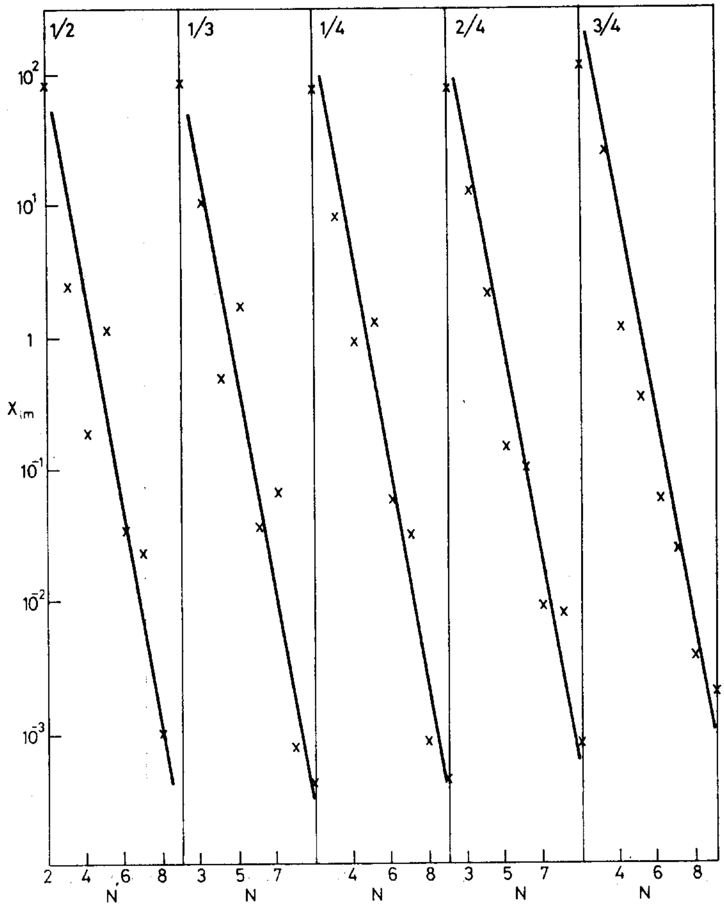

FIG. 3. The quantity $X_{\ell m}$ (see Sect. 2.4) against N. The interface is indicated as ℓ/m. The parallel straight lines are an indication of the exponential convergence rate.

So far we have concentrated only on the convergence of the solution at a set of points. An important feature of the GEM is the ability to implicitly satisfy the interface conditions. Figure 3 shows the results obtained for the discontinuity in the solution across the various interfaces $\Gamma_{\ell m}$. Here we have plotted the quantity

$$X_{\ell m} = \left\{\left(\int_{\Gamma_{\ell m}} | w_N^{(\ell)} - w_N^{(m)} |^2 \, ds\right)\right\}^{1/2}$$

Again, exponentially fast convergence is achieved, with the rate of convergence approximately the same at each interface. While the difference in solution values is decreasing, the solution on each side of the interface is tending to stabilise (as do the results in tables 2 and 3) and thus the GEM is successfully reproducing the interface conditions.

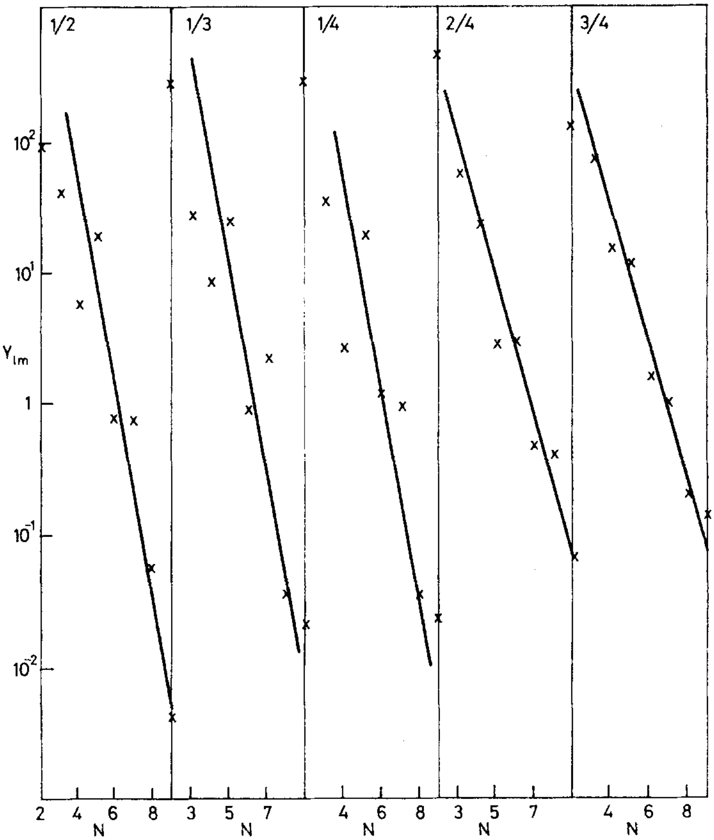

FIG. 4. The quantity $Y_{\ell m}$ (see Sect. 2.4) against N. The interface is indicated as ℓ/m. The straight lines are an indication of the exponential convergence rate.

Finally, figure 4 shows the results for the discontinuity of the normal derivatives.

$$Y_{\ell m} = \left\{ \int_{\Gamma_{\ell m}} \left(\frac{\partial w_N^{(\ell)}}{\partial n_\ell} + \frac{\partial w_N^{(m)}}{\partial n_m} \right)^2 ds \right\}^{1/2}$$

Note the addition sign in this latter expression. These results also demonstrate exponentially fast convergence, though at a rather slower rate than for $X_{\ell m}$.

From the viewpoint of the GEM, the results of Morley [11] are particularly interesting. These results were obtained using a FEM incorporating singular functions of the form (2.8). Morley makes the point that his converged results are probably sufficiently accurate for practical purposes (see Table 1, where the Morley results are accurate to about 0.3 %) but he classes them as unsatisfactory due to the behaviour of the coefficients associated with these singular functions. Table 3 shows the converged coefficients (suitably modified to take account of the differing size of his

TABLE 3

Values of the variational coefficients in R_1

	α_1	α_2	α_3	α_4
GEM ($N = 9$)	401.162	87.6558	17.2381	−8.07045
Rosser & Papamichael	401.1625	87.65592	17.23792	−8.071215
Morley	401.41	88.61	−20.07	−26.78

rectangle) together with the exact values and the final GEM values. There is clearly satisfactory agreement for the first coefficient. but not for the last two. Thus the cause of trouble in [11] is the poor determination of these higher coefficients, since in the method used there these higher singular functions are themselves capable of being approximated by the finite elements (see [5]). In the GEM this is not so since only one type of trial function is used near the singular point.

Griffiths [6] has also recently emphasised that the estimates of the coefficients of singular functions obtained from such a FEM calculation can depend critically on the detailed form of the singular functions.

4. Further Comments

In this paper we have considered the solution of a singular boundary value problem using the recently proposed GEM. We have demonstrated that the GEM is a feasible computational tool, and (at least for this problem) an exponentially fast convergence rate is achieved even in the neighbourhood of the singular point. From a comparison with the exact CTM results, the GEM yields very accurate results both for the solution and for the coefficients associated with the singular behaviour. This is particularly encouraging since none of the general numerical methods (finite differences or finite elements) referred to in Section 1 approach this accuracy (especially close to the singularity). Since the GEM is not restricted to the Laplacian operator used here, it may prove useful for determining accurate results for singular problems for which the CTM is not applicable.

We have not been primarily concerned with the efficiency of the method from the computational cost point of view. The programme used here had operation count for large N:

Setup matrices: $\mathcal{O}(N^5)$ approximately.
Solve equations: $\mathcal{O}(N^6)$

These counts are rather high. However, it is possible to achieve counts of $\mathcal{O}(N^4)$ for both phases of the work, this being achieved by careful choice of the trial functions (Chebyshev Polynomials) within the subregions; by using Fast Fourier techniques for the numerical quadratures; and by the use of an iterative technique to solve the linear equations [4].

These latter techniques have yet to be tested in practice; however, the experience with the GEM to date, although limited, is encouraging.

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